Local Time of Transient Random Walk on $\mathbb{Z}_+$

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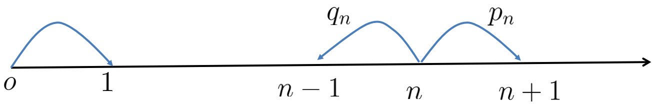
Contents

The Model

♢ **(1,1) Spatially Inhomogeneous Random Walks**

$$P(X_{k+1} = 1 | X_k = 0) = 1,$$
$$P(X_{k+1} = n+1 | X_k = n) = p_n,$$
$$P(X_{k+1} = n-1 | X_k = n) = q_n,$$



where $\forall n \geq 1,\ p_n, q_n > 0,\ p_n + q_n = 1.$

Transcience criterion

Let $\rho_n = q_n/p_n, n \geq 1$. Then the chain X is transient if and only if $\sum_{k=1}^{\infty} \rho_1 \cdots \rho_k < \infty$.

Cutpoints and Strong Cutpoints

♢ **Definition(Local Time)**

For $x \in \mathbb{Z}_+$, we call $\xi(x) = \sum_{k=0}^{\infty} 1_{\{X_k = x\}}$ the local time of the chain X at x.

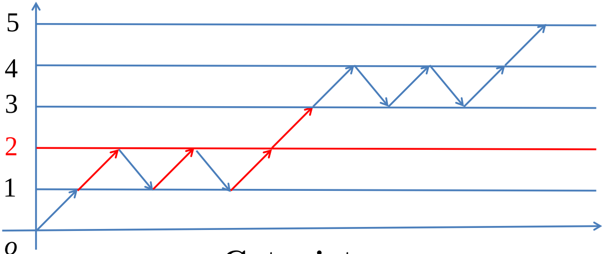
♢ **Definition(Cutpoint)**

A site R is a cutpoint if, for some k, we have $X_k = R$, and $\{X_0, X_1, ..., X_k\}$ is disjoint from $\{X_{k+1}, X_{k+2}, ...\}$, i.e., $X_i \leq R, i = 0, 1, ..., k$; $X_k = R$; and $X_i > R, i = k+1, k+2, ...$

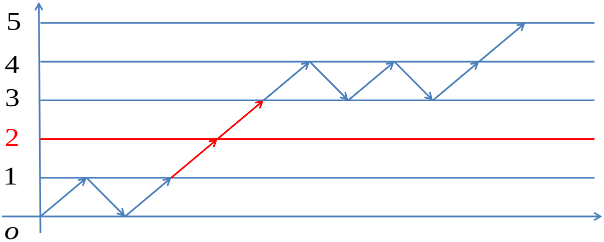
♢ **Definition(Strong Cutpoint)**

A site R is a strong cutpoint if, for some k, we have $X_k = R, X_i < R, i = 0, 1, ..., k-1$, and $X_i > R, i = k+1, k+2, ...$

♡ *Remark.* A site R is a strong cutpoint iff $\xi(R) = 1$.



Cutpoints



Strong Cutpoints

- ♡ For recurrent random walk, there is no cutpoint.
- ♡ For transient simple(spatially homogeneous) random walk, there must be infinitely many cutpoints.
- ♡ Intuitively, the **faster** the walk runs, the **more cutpoints** it has.
- ♡ Is it possible that the walk is **transient**, but not fast enough, so that there are only **finitely many cutpoints?**

- ◇ For the above model, James, Lyons, Peres (2008) give a example which shows that the walk is **transient** but has only **finitely** many cutpoints.
- ◇ Csáki, Földes, Révész (2010) give a **criterion** for the finiteness of the number of cutpoints.

E. Csáki, A. Földes, P. Révész, On the number of cutpoints of transient nearest neighbor random walk on the line, J. Theoret. Probab. 23 (2) (2010) 624-638.

N. James, R. Lyons, Y. Peres, A transient Markov chain with finitely many cutpoints, *In: IMS Collections Probability and Statistics: Essays in Honor of David A. Freedman*, **2** (2008) 24-29, Institute of Mathematical Statistics.

Set for $n \in \mathbb{Z}_+$,

$$\rho_n = \frac{q_n}{p_n} \text{ and } D(n) = 1 + \sum_{j=1}^{\infty} \rho_{n+1} \cdots \rho_{n+j}.$$

Theorem(Csáki, Földes, Révész (2010))

Suppose $0 \le p_i < 1/2, i \ge 1$.

- If
$$\sum_{n=1}^{\infty} \frac{1}{D(n) \log n} < \infty,$$
then almost surely, $\{X_n\}$ has **finitely** many cutpoints;
- If $\exists \delta > 0$ such that $D(n) \le \delta n \log n$ for n large enough and
$$\sum_{n=1}^{\infty} \frac{1}{D(n) \log n} = \infty,$$
then almost surely, $\{X_n\}$ has **infinitely** many strong cutpoints.

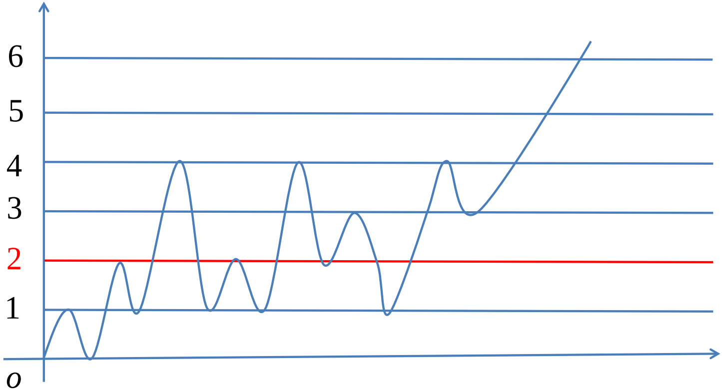
Some Questions

Csáki, Földes, Révész proposed the following open problem:

◇ **Open problem**(Csáki, Földes, Révész (2010)). Consider a spatially inhomogeneous random walk. For $x \in \mathbb{Z}_+$, let $\xi(x)$ be the local time of the walk at x. For any positive integer a, is there a criterion to determine whether the cardinality of the set $C := \{x \in \mathbb{Z}^+ : \xi(x) = a\}$ is finite or infinity?

Beside the above open problem, naturally, one may ask the following question:

◇ **Question.** Whenever there are infinitely many (strong) cutpoints, how many cutpoints are there in $[0, n]$?



$a = 8$ and $\xi(2) = a$,
the walk hits the site 2 exactly 8 times.

The Main Results

For any $a \in \mathbb{Z}_+$, set $C := \{x \in \mathbb{Z}^+ : \xi(x) = a\}$.

Theorem 1

Suppose that ρ_k is increasing in $k > N_0$ for some $N_0 > 0$ and $\rho_k \to 1$ as $k \to \infty$. If

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} < \infty,$$

then $|C| < \infty$ almost surely; if $D(n) \leq \delta n \log n$, $n \geq n_0$ for some $n_0 > 0$ and $\delta > 0$ and

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} = \infty,$$

then $|C| = \infty$ almost surely.

Theorem 1 gives an answer to the above mentioned open problem proposed by Csáki, Földes, Révész (2010).

A Concrete Example

Fix $\beta \in \mathbb{R}$. Clearly there exists $n_1 > 0$ such that $\frac{1}{4}\left(\frac{1}{n} + \frac{1}{n(\log\log n)^\beta}\right) \in (0, 1/2)$ for all $n > n_1$. For $n \geq 1$ set

$$r_n = \begin{cases} \frac{1}{4}\left(\frac{1}{n} + \frac{1}{n(\log\log n)^\beta}\right), & \text{if } n \geq n_1, \\ r_{n_1}, & \text{if } n < n_1. \end{cases} \quad (1)$$

Lemma 1

For $n \geq 1$ set $p_n = \frac{1}{2} + r_n$. Then $D(n) \sim n(\log\log n)^\beta$ as $n \to \infty$.

Corollary 1

For $n \geq 1$ set $p_n = \frac{1}{2} + r_n$. Then

$$\beta > 1 \Rightarrow |C| < \infty \text{ almost surely;}$$

$$\beta \leq 1 \Rightarrow |C| = \infty \text{ almost surely.}$$

Number of Strong Cutpoints in $[0, n]$

Proposition 1

For $n \geq 1$ let r_n be the one in (1) and set $p_n = \frac{1}{2} + r_n$. Then

$$\lim_{n\to\infty} \frac{2(\log\log n)^\beta}{\log n} E\,|C \cap [0,n]| = 1.$$

Inspired by Proposition 1, one may expect that

$$\frac{2(\log\log n)^\beta}{\log n}\,|C \cap [0,n]| \to 1$$

almost surely as $n \to \infty$. However this is not the case. The main reason is that the events $\{x \in C\}, x = 1, 2, ..., n$ are not independent. We have the following theorem.

Theorem 2

Suppose that $a = 1$. Let $r_1 = 1/4$ and $r_n = \frac{1}{2n}, n \geq 2$. Set $p_n = \frac{1}{2} + r_n$, for $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{2\,|C \cap [0,n]|}{\log n} \overset{\mathcal{D}}{=} S$$

where S is an exponentially distributed random variable with $P(S > t) = e^{-t}, t > 0$.

As remarked above, when $a = 1$, the set C is the collection of strong cutpoints. For cutpoints defined above, similar result holds. Let $\tilde{C}$ be the collection of **cutpoints**. Under the conditions of Theorem 2, we have

$$\lim_{n\to\infty} \frac{|\tilde{C} \cap [0,n]|}{\log n} \overset{\mathcal{D}}{=} S.$$

Ideas of Proofs

Idea of the Proof of Theorem 1

For any $a \in \mathbb{Z}_+$, set $C := \{x \in \mathbb{Z}^+ : \xi(x) = a\}$.

Theorem 1

Suppose that ρ_k is increasing in $k > N_0$ for some $N_0 > 0$ and $\rho_k \to 1$ as $k \to \infty$. If

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} < \infty,$$

then $|C| < \infty$ almost surely; if $D(n) \leq \delta n \log n$, $n \geq n_0$ for some $n_0 > 0$ and $\delta > 0$ and

$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} = \infty,$$

then $|C| = \infty$ almost surely.

Step 1: **Distribution of Local Times**

For $n > m \geq 0$, write

$$D(m,n) := 1 + \sum_{j=1}^{n-m-1} \rho_{m+1} \cdots \rho_{m+j}, D(m) := \lim_{n\to\infty} D(m,n).$$

Lemma 2

We have for $x \geq 0$ and $a \geq 1$,

$$P(\xi(x) = a) = \frac{p_x}{D(x)} \left(1 - \frac{p_x}{D(x)}\right)^{a-1},$$

and for $m \geq 2$, $1 \leq j_1 < j_2 < ... < j_m$,

$$P(\xi(j_1) = 1, \cdots, \xi(j_m) = 1) = \frac{p_{j_1} p_{j_2} \cdots p_{j_m}}{D(j_1, j_2) D(j_2, j_3) \cdots D(j_{m-1}, j_m) D(j_m)}.$$

Step 2: **Cardinalities of Certain Simplexes in $\mathbb{Z}_+^j$**

For positive integers $a \geq j \geq i \geq 1$ set

$$S(a,j) = \left\{ (a_1, ..., a_j) : a_k \in \mathbb{Z}_+/\{0\}, j \geq k \geq 1, \sum_{k=1}^{j} a_k = a \right\},$$

$$\tilde{S}(a,j) = \left\{ (a_1, ..., a_j) : a_k \in \mathbb{Z}_+, j \geq k \geq 1, a_j \geq 1, \sum_{k=1}^{j} a_k = a \right\},$$

$$\tilde{S}_i(a,j) = \left\{ (a_1, ..., a_j) \in \tilde{S}(a,j) : \sum_{k=1}^{j} 1_{a_k \neq 0} = i \right\}.$$

Lemma 3

For $a \geq j \geq i \geq 1$, we have $|S(a,j)| = \binom{a-1}{j-1}$ and $|\tilde{S}_i(a,j)| = \binom{j-1}{i-1}\binom{a-1}{i-1}$.

Step 3: **Joint Probability of $\{x \in C\}$ and $\{y \in C\}$**

Based on Lemma 2 and Lemma 3, we have

Lemma 4

For $1 \leq x < y < \infty$ and $a \geq 1$, we have

$$P(\xi(x) = a, \xi(y) = a)$$
$$= \sum_{i=1}^{a} \binom{a-1}{i-1}^2 \left(\frac{p_x}{D(x,y)}\right)^i \left(1 - \frac{p_x}{D(x,y)}\right)^{a-i}$$
$$\times \left(q_y \left(1 - \frac{D(x,y-1)}{D(x,y)}\right)\right)^{i-1}$$
$$\times \left(q_y \frac{D(x,y-1)}{D(x,y)} + p_y \left(1 - \frac{1}{D(y)}\right)\right)^{a-i} \frac{p_y}{D(y)}.$$

Step 4: **Dependence of $\{x \in C\}$ and $\{y \in C\}$**

Based on Lemma 4, we can show

Proposition 2

For each $\varepsilon > 0$, there exist $N > 0$ and $M > 0$ such that

$$1 - \varepsilon \leq \frac{D(x, x+y)}{D(x)} \frac{P(\xi(x) = a, \xi_{x+y} = a)}{P(\xi(x) = a)P(\xi(x+y) = a)} \leq 1 + \varepsilon,$$

for all $x > N, y > M$.

Based on Lemma 2 and Proposition 2, roughly speaking, Theorem 1 can be proved by **Borel-Cantelli** lemma. □

Idea of the Proof of Theorem 2

Theorem 2

Suppose that $a = 1$. Let $r_1 = 1/4$ and $r_n = \frac{1}{2n}, n \geq 2$. Set $p_n = \frac{1}{2} + r_n$, for $n \geq 1$. Then

$$\lim_{n\to\infty} \frac{2\,|C \cap [0, n]|}{\log n} \stackrel{\mathcal{D}}{=} S$$

where S is an exponentially distributed random variable with $P(S > t) = e^{-t}, t > 0$.

Theorem 2 is proved by an **moment method**. The key step is to show that the moments of $2\,|C \cap [0, n]|$ converge to those of exponential distribution, which is very complicated even if we consider here only the case $a = 1$ and $\beta = 0$.

Convergence of the Moments

Lemma 5

Under the conditions of Theorem 2, we have

$$\lim_{n\to\infty} \frac{E(|C \cap [0,n]|^k)}{(\log n)^k} = \frac{2^k}{k!}, \ k \geq 1.$$

Sketch of Proof.

For $k \geq 1$, we set $\eta_k = \begin{cases} 1, k \in C \\ 0 \ k \notin C \end{cases}$. Then $|C \cap [0,n]| = \sum_{k=0}^{n} \eta_k$.

Let $S(a,j)$ be the above defined simplex in $\mathbb{Z}_+^j$. Then

$$E|C \cap [0,n]|^k = E\left(\left(\sum_{j=0}^{n} \eta_j\right)^k\right)$$

$$= \sum_{1 \le j_1, j_2, \ldots, j_k \le n} E(\eta_{j_1} \eta_{j_2} \cdots \eta_{j_k})$$

$$= \sum_{m=1}^{k} \sum_{\substack{l_1 + \cdots + l_m = k, \\ l_i \ge 1, i = 1, \ldots, m}} m! \sum_{0 \le j_1 < \ldots < j_m \le n} E(\eta_{j_1}^{l_1} \cdots \eta_{j_m}^{l_m})$$

$$= \sum_{m=1}^{k} \sum_{(l_1, \ldots, l_m) \in S(k,m)} m! \sum_{0 \le j_1 < \ldots < j_m \le n} E(\eta_{j_1}^{l_1} \cdots \eta_{j_m}^{l_m}).$$

Using the above Lemma 2(local time distribution) and Lemma 3(Combinatorial result), we get

$$E|C \cap [0,n]|^k = \sum_{m=1}^{k} \binom{k-1}{m-1} m! \sum_{0 \le j_1 < \ldots < j_m \le n} E(\eta_{j_1} \cdots \eta_{j_m})$$

$$= \sum_{m=1}^{k} \binom{k-1}{m-1} m! \sum_{0 \le j_1 < \ldots < j_m \le n} \frac{p_{j_1} p_{j_2} \cdots p_{j_m}}{\prod_{s=1}^{m-1} D(j_s, j_{s+1}) D(j_m)}$$

$$=: \sum_{m=1}^{k} \binom{k-1}{m-1} m! G(n,m).$$

What is left for us to compute is

$$G(n,m) := \sum_{0 \le j_1 < \ldots < j_m \le n} \frac{p_{j_1} p_{j_2} \cdots p_{j_m}}{\prod_{s=1}^{m-1} D(j_s, j_{s+1}) D(j_m)}.$$

We need the follow estimations of $D(i)$ and $D(i,j)$, and another combinatorial result.

Lemma 6

Under the conditions of Theorem 2, there exists a number $i_0 > 0$ such that

$$(1-\varepsilon)\frac{i(j-i)}{j} \leq D(i,j) \leq (1+\varepsilon)\frac{i(j-i)}{j}, \ j > i \geq i_0,$$

$$(1-\varepsilon)i \leq D(i) \leq (1+\varepsilon)i, \ i \geq i_0.$$

Lemma 7

Fix $k \geq 1, l \geq 1$. We have

$$\lim_{n\to\infty} \frac{1}{(\log n)^k} \sum_{l\leq j_1<\ldots<j_k\leq n} \frac{1}{j_1(j_2-j_1)\cdots(j_k-j_{k-1})} = 1.$$

With the help of Lemma 6 and Lemma 7, we can show

$$\lim_{n\to\infty} \frac{G(n,m)}{(\log n)^m} = \frac{1}{2^m}$$

which leads to

$$\lim_{n\to\infty} \frac{E|C\cap[0,n]|^k}{(\log n)^k} = \frac{k!}{2^k}.$$

□

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